J. Fluid Mech. (1968), *vol.* 34, *part* 3, *pp.* 531–549

Printed in Great Britain

Resonant gravity-wave interactions in a shear flow

By ALEX. D. D. CRAIK

Department of Applied Mathematics, University of St Andrews, Fife, Scotland

(Received 22 March 1968)

Among a triad of gravity waves in a uniform shear flow, a remarkably powerful second-order resonant interaction may take place. This interaction is characterized by large growth rates of waves which propagate in directions oblique to that of the primary flow, and by a systematic transfer of energy from the primary flow to such waves. Most of the energy transfer takes place in the vicinity of a 'critical layer', where viscous forces are dominant.

Provided the resonance condition may be satisfied, a uniform shear flow which is perturbed by a two-dimensional wave of small but finite amplitude may be unstable, owing to the growth of two initially infinitesimal oblique waves which complete the resonant triad.

1. Introduction

Phillips (1960) has shown that no second-order resonant interactions are possible among a triad of gravity waves in a liquid which is otherwise at rest, but that third-order resonant interactions can occur among a group of four such waves. Further work on gravity waves by Benjamin & Fier (1967) and on internal waves by Davis & Acrivos (1968) reveals that a single wave of finite amplitude may be unstable, in the sense that it becomes distorted by the growth of resonant harmonics which feed upon its energy. Such interactions are manifestations of an energy-sharing mechanism in which, if viscous dissipation is neglected, the total amount of wave energy is conserved.

It is here shown that the presence of a uniform shear flow in the liquid can permit a remarkably strong second-order resonant interaction among three suitable gravity waves, and that this resonance can produce continuous wave growth due to systematic extraction of energy from the primary flow. In this case, the total wave energy is no longer conserved, but increases with time. One consequence of this result is that, in the presence of a suitable wave of small but finite amplitude, two initially infinitesimal waves which complete the resonant triad may grow indefinitely (or, rather, until the second-order theory is inapplicable), while the original wave remains essentially unaltered.

Recently, Kelly (1968) has examined second-order resonant interactions of disturbances in two particular inviscid shear flows. For one of these flows, a stably stratified antisymmetric shear layer, resonance leads to the temporal growth of a disturbance with fixed spatial periodicity. In this case, a net energy transfer from the primary flow to the disturbance does indeed take place.

Kelly's analysis deals only with two-dimensional disturbances which propagate in the direction of the mean flow. In contrast, three-dimensionality is an essential feature of the present analysis; for, unlike the cases examined by Kelly, no resonant interaction is found to be possible among three gravity waves which propagate in the direction of flow.

Raetz (1959) has demonstrated that certain three-dimensional disturbances in Blasius flow fulfil the conditions of second-order resonance, but it is not clear whether this resonance can lead to energy-extraction from the primary flow or simply to energy-sharing among the waves. Stuart (1962) and Benney (1961, 1964) have also examined three-dimensional disturbances of finite amplitude in parallel flows, but none of these papers specifically concerns resonance phenomena.

The present work examines second-order resonant interactions of gravity waves in a liquid of small viscosity, which possesses a linear velocity profile. The results are subsequently extended to waves on the interface between two fluids of different densities, each of which may have a linear velocity profile. The restriction to linear velocity profiles is made for convenience, but it seems likely that similar results might hold for curved profiles. That the chosen velocity becomes indefinitely large at great depths is no disadvantage; for the only relevant portion of the velocity profile is that within a layer near the free surface, the depth of which is comparable to the wavelengths of the disturbances considered.

Not all possible resonant triads are examined. For simplicity, attention is restricted to those which are composed of one two-dimensional wave propagating in the direction of the primary flow, and two oblique waves which propagate at equal and opposite angles to this direction. For such a triad, resonance occurs when (i) the wave-number components of the oblique waves in the direction of the primary flow are equal to half the wave-number of the two-dimensional wave, and (ii) the components of all three phase velocities in the direction of the primary flow are equal. With a given two-dimensional wave and a sufficiently large velocity gradient in the primary flow, these conditions are shown to be satisfied by two possible pairs of oblique waves, each of which has an angle of propagation between 60° and 90° to the direction of mean flow.

For gravity waves, the resonance condition turns out to be rather severe; mean velocity gradients being required which are probably in excess of any which might occur in nature (for example, in wind-driven surface currents in the ocean). However, for interfacial waves in liquids with a small density difference the condition for resonance is more easily achieved, provided the velocity gradients in the two liquids are not equal.

The essential features of the analysis are as follows. At resonance, all three waves have the same 'critical layer' at which the component of phase velocity in the direction of mean motion equals the velocity of the primary flow. Except near this critical layer (and in a weak boundary layer near the free surface, whose effect is slight) an inviscid analysis is valid. For the oblique waves, linearized inviscid theory yields singularities at the critical layer in the horizontal velocity components which are at right angles to the respective directions of propagation.

These 'cross velocities' are a direct consequence of the periodic stretching and contraction of vortex lines associated with the primary flow. When pursued to second order, the inviscid analysis exhibits strong (at worst, fourth-order) singularities at the critical layer. Consideration of viscous effects near the critical layer removes these singularities, and the resultant expressions for the second-order growth rates of the oblique waves are found to be dominated by the contribution of these viscous terms. More precisely, if the free-surface displacements associated with the oblique waves are $\mathrm{Re}\{a_{1,2}(t)\exp[i(\frac{1}{2}kx \pm ly - \frac{1}{2}\omega t)]\}$, and that due to the two-dimensional wave is $\mathrm{Re}\{a_3(t)\exp[i(kx-\omega t)]\}$, where k, l and ω are real, the complex wave-amplitudes $a_{1,2,3}$ are found to satisfy equations of the form

$$\frac{da_1}{dt} = O\left(\frac{\omega^2}{k\nu}\right) a_2^* a_3, \quad \frac{da_2}{dt} = O\left(\frac{\omega^2}{k\nu}\right) a_1^* a_3, \quad \frac{da_3}{dt} = O(\omega k) a_1 a_2,$$

where * denotes the complex conjugate and ν is the kinematic viscosity of the liquid.

The surprising feature of these results is that the growth rates of the oblique waves are proportional to ν^{-1}, and are consequently much larger than expected. This indicates a very strong resonant interaction in which viscous forces near the critical layer transfer energy from the primary flow to the waves (or *vice versa*). If the initial phases of $a_{1,2,3}$ are appropriately chosen, the oblique waves grow continuously while the two-dimensional wave is little changed.

Because the resonant interaction is so strong, this second-order analysis holds only for rather small wave amplitudes. In particular, the usual assumption that the wave slopes are small compared with unity does not here ensure that the growth rates are small compared with the frequencies of the waves. Instead, the necessary requirement is that a Reynolds number based upon wave-amplitude and phase velocity should be small compared with unity.

2. Inviscid linear theory

The flow configuration is shown in figure 1. The mean velocity increases linearly with vertical depth z below the undisturbed liquid surface, the x-axis is taken horizontally in the direction of motion, and the y-axis (not shown) completes the right-handed Cartesian system. Velocity components in the x-, y- and z-directions are denoted by u, v and w respectively. The primary flow is

$$u \equiv \bar{u} = \bar{u}'z, \quad v = w = 0;$$

and, superimposed on this flow, there are three small travelling-wave disturbances with (x,t)-dependence of the form

$$\exp\{i(\tfrac{1}{2}kx + ly - \omega_0 t)\}, \quad \exp\{i(\tfrac{1}{2}kx - ly - \omega_0 t)\}, \quad \exp\{i(kx - \omega t)\},$$

which, for convenience, will be called waves 1, 2 and 3 respectively. The constants k, l, ω_0 and ω are assumed to be real.

The velocity perturbations associated with wave 3 are

$$\left.\begin{aligned} u_3 &= -iA_3\exp\{-kz\}\exp\{i(kx-\omega t)\}, \quad v_3 = 0, \\ w_3 &= A_3\exp\{-kz\}\exp\{i(kx-\omega t)\}, \end{aligned}\right\} \tag{2.1}$$

where A_3 is some complex constant, and it is understood that only the real parts of these expressions have physical significance.

On introducing new horizontal co-ordinates $\hat{x}_1$, $\hat{y}_1$ defined as

$$m\hat{x}_1 = \tfrac{1}{2}kx + ly, \quad m\hat{y}_1 = -lx + \tfrac{1}{2}ky,$$

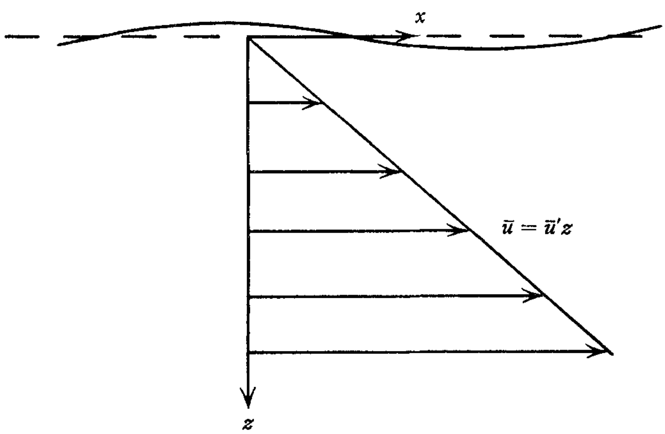

FIGURE 1. The flow configuration.

where $m = (\frac{1}{4}k^2 + l^2)^{\frac{1}{2}}$, the velocity components $\hat{u}_1, \hat{v}_1, w_1$ of wave 1 in the directions $\hat{x}_1$, $\hat{y}_1$ and z are found to be

$$\left.\begin{aligned} \hat{u}_1 &= -iA_1 \exp\{-mz\} \exp\{i(m\hat{x}_1 - \omega_0 t)\}, \quad w_1 = A_1 \exp\{-mz\} \exp\{i(m\hat{x}_1 - \omega_0 t)\}, \\ \hat{v}_1 &= \frac{2l}{imk(z-\alpha)} A_1 \exp\{-mz\} \exp\{i(m\hat{x}_1 - \omega_0 t)\}, \end{aligned}\right\} \tag{2.2}$$

where $\alpha \equiv 2\omega_0/\bar{u}'k$. The expression for the 'cross-velocity' $\hat{v}_1$ is derived from the linearized equation of motion

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}k}{2m}\frac{\partial}{\partial \hat{x}_1}\right)\hat{v}_1 - w_1\left(\frac{\bar{u}'l}{m}\right) = 0,$$

and it is singular at the critical layer $z = \alpha$, where the fluid velocity equals the component of phase velocity in the x-direction. The corresponding velocity components u_1, v_1 in the x- and y-directions are given by

$$mu_1 = \tfrac{1}{2}k\hat{u}_1 - l\hat{v}_1, \quad mv_1 = l\hat{u}_1 + \tfrac{1}{2}k\hat{v}_1.$$

The velocity components u_2, v_2, w_2 associated with wave 2 may be obtained similarly, by writing $-l$ for l and A_2 for A_1 in the above results. The corresponding 'cross-velocity' $\hat{v}_2$ is in the direction $\hat{y}_2 = (l/m)x + (k/2m)y$, and the wave propagates in the direction $\hat{x}_2 = (k/2m)x - (l/m)y$.

At the free surface,

$$z = \zeta \equiv \zeta_1 + \zeta_2 + \zeta_3,$$

where ζ_i is the downwards vertical displacement of the surface due to the ith wave. The linearized kinematic conditions are therefore

$$\frac{\partial \zeta_i}{\partial t} = [w_i]_{z=0} \quad (i = 1, 2, 3).$$

Also, the condition that the pressure is constant at $z = \zeta$ gives the linearized result

$$[p_i]_{z=0} = -\rho g \zeta_i \quad (i = 1, 2, 3),$$

where p_i denotes the pressure fluctuation associated with the ith wave, ρ is the liquid density and g is gravitational acceleration.

For wave 3, the linearized x-momentum equation yields

$$\left[ikp_3 + \frac{\partial u_3}{\partial t} + \overline{u}' w_3\right]_{z=0} = 0,$$

which, together with the above results, leads to the equation

$$\left[\frac{\partial^3 w_3}{\partial t^2\, \partial z} - ik\overline{u}' \frac{\partial w_3}{\partial t} - gk^2 w_3\right]_{z=0} = 0. \tag{2.3a}$$

Similarly, for wave 1, the equation of motion in the $\hat{x}_1$-direction gives the result

$$\left[\frac{\partial^3 w_1}{\partial t^2\, \partial z} - \tfrac{1}{2} ik\overline{u}' \frac{\partial w_1}{\partial t} - gm^2 w_1\right]_{z=0} = 0; \tag{2.3b}$$

and, for wave 2, an equation identical with this is satisfied by w_2. Equations (2.3*a*, *b*) yield the dispersion relations for ω and ω_0, namely

$$\omega^2 - \overline{u}'\omega - gk = 0, \quad \omega_0^2 - \frac{k\overline{u}'}{2m}\omega_0 - gm = 0. \tag{2.4a, b}$$

3. The resonance condition

The condition that waves 1, 2 and 3 should form a resonant triad is that

$$\omega = 2\omega_0. \tag{3.1}$$

We must now examine whether, for given $\overline{u}'$ and k, there exist permissible values of m such that this condition is satisfied. Since $m = (\frac{1}{4}k^2 + l^2)^{\frac{1}{2}}$, it is necessary that m/k should be greater than $\frac{1}{2}$.

Without loss of generality, we may assume that k, m, ω and ω_0 are all positive, but that $\overline{u}'$ may be positive or negative. Then, from equations (2.4*a*, *b*) and (3.1), we require that

$$(\overline{u}'^2 + 4gk)^{\frac{1}{2}} + \overline{u}' = [(k\overline{u}'/m)^2 + 16gm]^{\frac{1}{2}} + k\overline{u}'/m;$$

and this equation may be re-expressed as a quadratic in m/k, namely

$$8(m/k)^2 - (\lambda + 2)(m/k) + \lambda = 0,$$

where

$$\lambda = \frac{\overline{u}'}{(gk)^{\frac{1}{2}}}\left[\left(\frac{\overline{u}'^2}{gk} + 4\right)^{\frac{1}{2}} + \frac{\overline{u}'}{(gk)^{\frac{1}{2}}}\right].$$

When $\overline{u}'$ is negative, there is no real root m/k which is greater than $\frac{1}{2}$. Accordingly, there cannot be resonance for $\overline{u}' < 0$. However, there are two real roots m/k which are greater than $\frac{1}{2}$ when $\lambda \geqslant 2(7+48^{\frac{1}{2}})$, and this condition is satisfied whenever

$$\frac{\overline{u}'}{(gk)^{\frac{1}{2}}} \geqslant \frac{7+48^{\frac{1}{2}}}{(8+48^{\frac{1}{2}})^{\frac{1}{2}}}. \tag{3.2}$$

Therefore, when $\overline{u}'(gk)^{-\frac{1}{2}}$ is sufficiently large, there exist two pairs of oblique waves which form a resonant triad with a two-dimensional wave of wavenumber k. The angle of propagation of these waves to the x-axis is $\theta = \cos^{-1}(k/2m)$ and it is readily shown that all permissible values of θ lie between 60° and 90°. In figure 2, the resonance condition is displayed as a graph of θ against $\overline{u}'(gk)^{-\frac{1}{2}}$. At the minimum value of 3·60 for $\overline{u}'(gk)^{-\frac{1}{2}}$, θ is 74° 27′; and, when $\overline{u}'(gk)^{-\frac{1}{2}}$ is large, the two values of θ approach 60° and 90°.

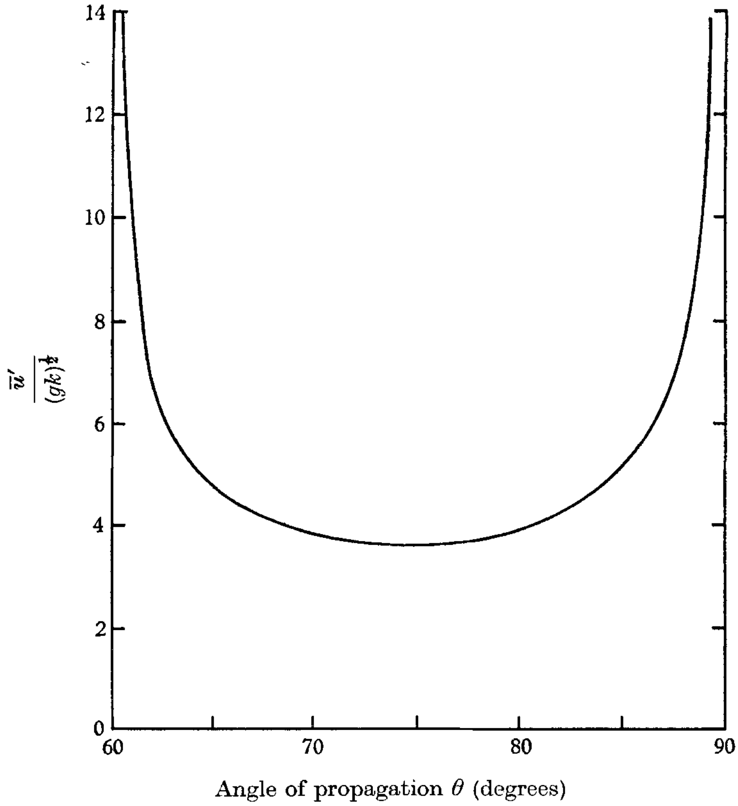

FIGURE 2. The resonance condition.

4. Viscous linear theory near $z = \alpha$

For oblique waves, the linear inviscid theory yields singular solutions at $z = \alpha$. As shown in § 2, these singularities occur in the 'cross-velocity' $\hat{v}_1$ associated with wave 1 and in the corresponding velocity component of wave 2. These singularities may be eliminated by including viscous terms in the vicinity of $z = \alpha$. This has been done by Benney (1961, 1964); and, for later reference, the results are summarized here.

On retaining the highest-order terms in $(\bar{u}'/k^2\nu)$ near $z = \alpha$, the (viscous) linearized equation for $\hat{v}_1$ reduces to

$$\left(\frac{d^2}{dZ^2}+Z\right)L(Z) = 1, \tag{4.1}$$

$$\hat{v}_1 = \frac{2l}{m}\left(\frac{\bar{u}'}{2k^2\nu}\right)^{\frac{1}{3}} L(Z)A_1 \exp\{-m\alpha\}\exp\{i(m\hat{x}_1-\omega_0 t)\}, \quad Z = i\left(\frac{\bar{u}'k}{2\nu}\right)^{\frac{1}{3}}(z-\alpha),$$

where $(\bar{u}'k/2\nu)^{\frac{1}{3}}$ is large compared with k. Also, $L(Z)$ satisfies the boundary conditions $L(Z)\to Z^{-1}$ as $Z\to \pm i\infty$. By symmetry, similar results hold for $\hat{v}_2$. The solution $L(Z)$ of (4.1) is a Lommel function, several properties of which are described by Benney (1961). In the present paper, the following properties will be used.

(i) When $Z = iY$ and Y is real,

$$L(Z) = L_r(Y)+iL_i(Y),$$

where L_r is an even and L_i an odd function of Y. Also, L_r and L_i satisfy the equations

$$\frac{d^2L_r}{dY^2}+YL_i = -1, \quad \frac{d^2L_i}{dY^2}-YL_r = 0. \tag{4.2a, b}$$

(ii) $L(Z)$ may be expressed in terms of solutions h_1, h_2 of the homogeneous equation $d^2h/dZ^2+Zh = 0$, as

$$L(Z) = \left\{h_2(Z)\int_{i\infty}^{Z} h_1(\zeta)\,d\zeta - h_1(Z)\int_{-i\infty}^{Z} h_2(\zeta)\,d\zeta\right\}\bigg/ W\{h_1(Z),h_2(Z)\}, \tag{4.3}$$

provided $-\frac{2}{3}\pi < \arg(Z) < \frac{2}{3}\pi$. The solutions h_1, h_2 are the modified Hankel functions

$$h_1(Z) = (\tfrac{2}{3}Z^{\frac{3}{2}})^{\frac{1}{3}}H^{(1)}_{\frac{1}{3}}(\tfrac{2}{3}Z^{\frac{3}{2}}), \quad h_2(Z) = (\tfrac{2}{3}Z^{\frac{3}{2}})^{\frac{1}{3}}H^{(2)}_{\frac{1}{3}}(\tfrac{2}{3}Z^{\frac{3}{2}}),$$

and $W\{h_1(Z),h_2(Z)\}$ is the Wronskian of $h_1(Z)$ and $h_2(Z)$, which is found to satisfy

$$W\{h_1(Z),h_2(Z)\} = -\frac{4i}{\pi}\left(\frac{3}{2}\right)^{\frac{1}{3}}. \tag{4.4}$$

For later use we also require an expression for the velocity component $\hat{v}_1^*$ in the $\hat{y}_1$-direction which is associated with the 'complex-conjugate' wave with $(\hat{x}_1,t)$-dependence $\exp\{-i(m\hat{x}_1-\omega_0 t)\}$. (We recall that the velocity component in the $\hat{y}_1$ direction is actually $\mathrm{Re}\{\hat{v}_1\} = \frac{1}{2}\{\hat{v}_1+\hat{v}_1^*\}$.) For pure imaginary values of Z, the appropriate expression is

$$\hat{v}_1^* = \frac{2l}{m}\left(\frac{\bar{u}'}{2k^2\nu}\right)^{\frac{1}{3}} L(-Z)A_1^* \exp\{-m\alpha\}\exp\{-i(m\hat{x}_1-\omega_0 t)\}, \tag{4.5}$$

where A_1^* is the complex conjugate of A_1, and $L(-Z)$ equals $L_r(Y)-iL_i(Y)$ by virtue of (i). A similar expression may be derived for the corresponding velocity component $\hat{v}_2^*$ of wave 2.

5. The inviscid second-order equations

At resonance, the interaction of waves 1 and 2 produces some inertia terms whose (x,t)-dependence is identical with that of wave 3. The second-order equations of motion for wave 3 are therefore of the form

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u_3+\bar{u}'w_3+\frac{1}{\rho}\frac{\partial p_3}{\partial x}=-[\mathbf{u}.\nabla u],$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)v_3=-[\mathbf{u}.\nabla v],$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)w_3+\frac{1}{\rho}\frac{\partial p_3}{\partial z}=-[\mathbf{u}.\nabla w],$$

$$\frac{\partial u_3}{\partial x}+\frac{\partial w_3}{\partial z}=0,$$

where the results of linear theory are used in evaluating the right-hand sides, and only those terms are retained which have (x,t)-dependence like $\exp\{i(kx-\omega t)\}$. Some care must be taken in deriving the non-linear terms. In particular, use must be made of the multiplication rule $\mathrm{Re}\{A\}\mathrm{Re}\{B\}=\frac{1}{2}\mathrm{Re}\{AB+AB^*\}$, where B^* is the complex conjugate of B. Assuming that the x-dependence of wave 3 remains like $\exp\{ikx\}$, we may eliminate u_3 and p_3 from the above equations to obtain

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)(w_3''-k^2w_3)=k^2[\mathbf{u}.\nabla w]+ik\frac{\partial}{\partial z}[\mathbf{u}.\nabla u],$$

where the primes denote differentiation with respect to z. Also, the relevant inertia terms may be expressed in the form

$$[\mathbf{u}.\nabla w]=-\frac{2l^2}{m}w_1w_2+\frac{ikl}{2m}(\hat{v}_2w_1-\hat{v}_1w_2),$$

$$[\mathbf{u}.\nabla u]=\frac{ikl^2}{m^2}(w_1w_2-\hat{v}_1\hat{v}_2)+\frac{l}{2m}\left(\frac{\partial}{\partial z}+\frac{k^2}{m}\right)(w_1\hat{v}_2-w_2\hat{v}_1),$$

if it is assumed that the z-dependence of w_1 and w_2 is that given by inviscid theory. (No assumption has been made regarding the z-dependence of $\hat{v}_1$ and $\hat{v}_2$: accordingly, the above expressions remain applicable near $z=\alpha$ on using the viscous solutions for $\hat{v}_1$ and $\hat{v}_2$.) The final second-order inviscid equation for w_3 is then

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)(w_3''-k^2w_3)=\frac{k^2l^2}{m^2}\frac{\partial}{\partial z}(\hat{v}_1\hat{v}_2)+\frac{ikl}{2m}\left(\frac{\partial^2}{\partial z^2}+\frac{k^2}{m}\frac{\partial}{\partial z}+k^2\right)(w_1\hat{v}_2-w_2\hat{v}_1) \quad (5.1a)$$

$$=\frac{-4l^2}{m^2}(z-\alpha)^{-1}\left\{(z-\alpha)^{-2}+\frac{2l^2}{m^2}[m+(z-\alpha)^{-1}]^2\right\}w_1w_2, \quad (5.1b)$$

where the latter, but not the former, version of the right-hand side is obtained on using the estimates (2.2) of inviscid theory for $\hat{v}_1$ and $\hat{v}_2$. Clearly, the inviscid analysis leads to a strong singularity at $z=\alpha$.

The second-order equations for waves 1 and 2 are found similarly. For wave 2, the results are

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)(w_2''-m^2w_2) = \frac{\partial}{\partial z}[\mathbf{u}\,.\,\nabla(\tfrac{1}{2}iku-ilv)]+m^2[\mathbf{u}\,.\,\nabla w],$$

where $$[\mathbf{u}\,.\,\nabla(\tfrac{1}{2}iku-ilv)] = -\tfrac{1}{4}m(2m+k)(w_3w_1^*)-\frac{ikl}{2m}\left(\frac{\partial}{\partial z}+k\right)(w_3\hat{v}_1^*),$$

$$[\mathbf{u}\,.\,\nabla w] = -\frac{(k+m)(k+2m)}{4m}(w_3w_1^*)-\frac{ikl}{2m}(w_3\hat{v}_1^*).$$

The equation for w_2 is then

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)(w_2''-m^2w_2) = -\frac{ikl}{2m}\left(\frac{\partial^2}{\partial z^2}+k\frac{\partial}{\partial z}+m^2\right)(w_3\hat{v}_1^*) \tag{5.2a}$$

$$= \frac{l^2}{m^2}(z-\alpha)^{-1}\{2(z-\alpha)^{-2}+(2m+k)[m+(z-\alpha)^{-1}]\}w_3w_1^*, \tag{5.2b}$$

where the inviscid estimate of $\hat{v}_1^*$ has been used to obtain (5.2*b*).

The equation for w_1 is found by symmetry, on interchanging the subscripts 1 and 2 and writing $-l$ for l.

6. The second-order solution

For each wave, w_i consists of an irrotational part W_i and a rotational part $\mathscr{W}_i$, where

$$\nabla^2 W_i = 0$$

and $\mathscr{W}_i$ is a particular solution of the appropriate non-homogeneous second-order equation. At large depths, both W_i and $\mathscr{W}_i$ must tend to zero. Since $\mathscr{W}_i$ is not a solution of the linearized equations, it must be a second-order small quantity. We may therefore assume that W_i is large compared with $\mathscr{W}_i$. Accordingly, to the present approximation, it is sufficient to allow for changes in amplitude of the *irrotational* parts W_i, while treating the *rotational* parts $\mathscr{W}_i$ as periodic in time. The non-homogeneous equations may then be used to determine the periodic components $\mathscr{W}_i$ (to within an arbitrary additive term ϵW_i, where ϵ is small), and the free-surface boundary condition yields the time-dependence of W_i.

We write $w_3 = W_3+\mathscr{W}_3$ and $w_2 = W_2+\mathscr{W}_2$, where

$$W_3 = A_3(t)\exp\{-kz\}\exp\{i(kx-\omega t)\}, \qquad \mathscr{W}_3 = \tilde{\mathscr{W}}_3(z)\exp\{i(kx-\omega t)\},$$

$$W_2 = A_2(t)\exp\{-mz\}\exp\{i(m\hat{x}_2-\tfrac{1}{2}\omega t)\}, \quad \mathscr{W}_2 = \tilde{\mathscr{W}}_2(z)\exp\{i(m\hat{x}_2-\tfrac{1}{2}\omega t)\};$$

also, w_1 may be represented similarly to w_2. We further assume that the rate of change of $A_i(t)$ is sufficiently small that

$$|\dot{A}_i/\omega A_i| \ll 1 \quad (i = 1,2,3), \tag{6.1}$$

where the dot denotes d/dt.

The second-order free-surface boundary conditions for waves 1, 2 and 3 are similar to the linearized equations (2.3a, b), but the right-hand sides are now no longer zero. Instead, they contain terms which derive from the inertia of the liquid at the free surface. The left-hand sides of these equations may be written as

$$\left[\frac{\partial^3 w_3}{\partial t^2 \partial z} - ik\bar{u}'\frac{\partial w_3}{\partial t} - gk^2 w_3\right]_{z=0} = \{ik(2\omega - \bar{u}')\dot{A}_3 - \omega^2(\tilde{\mathscr{W}}_3' + k\tilde{\mathscr{W}}_3)_{z=0}\}\exp\{i(kx-\omega t)\}, \quad (6.2a)$$

$$\left[\frac{\partial^3 w_j}{\partial t^2 \partial z} - \tfrac{1}{2}ik\bar{u}'\frac{\partial w_j}{\partial t} - gm^2 w_j\right]_{z=0}$$
$$= \{i(m\omega - \tfrac{1}{2}k\bar{u}')\dot{A}_j - \tfrac{1}{4}\omega^2(\tilde{\mathscr{W}}_j' + m\tilde{\mathscr{W}}_j)_{z=0}\}\exp\{i(m\hat{x}_j - \tfrac{1}{2}\omega t)\} \quad (j = 1, 2), \quad (6.2b)$$

on using the dispersion relations (2.4a, b). Also, the inertia terms are $O(\omega k^2 A_1 A_2)$ in the boundary condition for w_3, and $O(\omega k^2 A_3 A_1^*)$ in that for w_2. In fact, a precise derivation of all these terms is unnecessary, since it turns out that the contributions due to $(\tilde{\mathscr{W}}_j' + m\tilde{\mathscr{W}}_j)$ $(j = 1, 2)$ play a dominant role in determining the wave growth.

Now,
$$[\tilde{\mathscr{W}}_3' + k\tilde{\mathscr{W}}_3]_{z=0} = -\int_0^\infty (\tilde{\mathscr{W}}_3'' - k^2\tilde{\mathscr{W}}_3)\, e^{-kz}\, dz \quad (6.3a)$$

and
$$[\tilde{\mathscr{W}}_j' + m\tilde{\mathscr{W}}_j]_{z=0} = -\int_0^\infty (\tilde{\mathscr{W}}_j'' - m^2\tilde{\mathscr{W}}_j)\, e^{-mz}\, dz \quad (j = 1, 2), \quad (6.3b)$$

since the $\tilde{\mathscr{W}}_i$ $(i = 1, 2, 3)$ decay to zero at large z. Also, since the $\mathscr{W}_i$ are assumed to be periodic in x and t, the above integrands may be evaluated by means of the non-homogeneous equations of motion (5.1) and (5.2). However, the *inviscid* equations (5.1b) and (5.2b) yield expressions for these integrands which are singular at the critical layer $z = \alpha$, the strongest singularity being of fourth order. The inviscid equations are therefore inadequate for the present purpose, and the influence of viscosity near $z = \alpha$ must be examined.

7. Viscous second-order theory near $z = \alpha$

The viscous second-order equations satisfied by $\mathscr{W}_3$ and $\mathscr{W}_2$ are similar to (5.1a) and (5.2a), but include, respectively, the viscous terms

$$\nu(\mathscr{W}_3^{\text{iv}} - 2k^2\mathscr{W}_3'' + k^4\mathscr{W}_3), \quad \nu(\mathscr{W}_2^{\text{iv}} - 2m^2\mathscr{W}_2'' + m^4\mathscr{W}_2)$$

on the left-hand sides. It is convenient to define the new variable

$$Z_3 = i(\bar{u}'k/\nu)^{\frac{1}{3}}(z-\alpha),$$

and to reintroduce the variable $Z = i(\bar{u}'k/2\nu)^{\frac{1}{3}}(z-\alpha)$ which was used in § 4. From results (2.1), (4.1) and (4.5), we find that, near $z = \alpha$,

$$\hat{v}_1\hat{v}_2 = \frac{-4l^2}{m^2}\left(\frac{\bar{u}'}{2k^2\nu}\right)^{\frac{2}{3}}\{L(Z)\}^2 A_1 A_2 \exp\{-2m\alpha\}\exp\{i(kx-\omega t)\},$$

$$w_1\hat{v}_2 - w_2\hat{v}_1 = -\frac{4l}{m}\left(\frac{\bar{u}'}{2k^2\nu}\right)^{\frac{1}{3}} L(Z)\, A_1 A_2 \exp\{-2m\alpha\}\exp\{i(kx-\omega t)\},$$

$$w_3\hat{v}_1^* = \frac{2l}{m}\left(\frac{\bar{u}'}{2k^2\nu}\right)^{\frac{1}{3}} L(-Z)\, A_3 A_1^* \exp\{-(m+k)\alpha\}\exp\{i(\tfrac{1}{2}kx - ly - \tfrac{1}{2}\omega t)\},$$

where only the highest powers of $(\bar{u}'/k^2\nu)$ have been retained.

Near $z = \alpha$, the equations for $\mathscr{W}_3$ and $\mathscr{W}_2$ reduce to

$$\left(\frac{d^2}{dZ_3^2}+Z_3\right)\tilde{\mathscr{W}}_3'' = F_3(Z_3), \quad \left(\frac{d^2}{dZ^2}+Z\right)\tilde{\mathscr{W}}_2'' = F_2(Z),$$

where

$$F_3(Z_3) = \frac{il^2}{m^2\nu}\left(\frac{k\bar{u}'}{\nu}\right)^{\frac{1}{3}} A_1A_2\exp\{-2m\alpha\}\left\{\frac{d^2}{dZ^2}[L(Z)]-\frac{4l^2}{m^2}L(Z)\frac{d}{dZ}[L(Z)]\right\},$$

$$Z = 2^{-\frac{1}{3}}Z_3,$$

and
$$F_2(Z) = \frac{il^2}{m^2\nu}\left(\frac{k\bar{u}'}{2\nu}\right)^{\frac{1}{3}} A_3A_1^*\exp\{-(m+k)\alpha\}\frac{d^2}{dZ^2}[L(-Z)].$$

Also, in order to match the inviscid solutions, $\tilde{\mathscr{W}}_i''$ must decay as Z^{-4} away from the critical layer. Near $z = \alpha$, the appropriate solution for $\tilde{\mathscr{W}}_3''$ is (cf. 4.3)

$$\tilde{\mathscr{W}}_3'' = \left\{h_2(Z_3)\int_{i\infty}^{Z_3} h_1(\zeta)F_3(\zeta)d\zeta - h_1(Z_3)\int_{-i\infty}^{Z_3} h_2(\zeta)F_3(\zeta)d\zeta\right\}\Big/ W\{h_1(Z_3), h_2(Z_3)\}$$
$$(-\tfrac{2}{3}\pi < \arg Z_3 < \tfrac{2}{3}\pi), \quad (7.1)$$

and a similar solution exists for $\tilde{\mathscr{W}}_2''$ with Z_3 replaced by Z and $F_3(\zeta)$ by $F_2(\zeta)$.

Having found these solutions, we may now evaluate the contributions to the integrals of (6.3*a*, *b*) which derive from the vicinity of the critical layer $z = \alpha$. Clearly, $|\mathscr{W}_i''| \gg k^2|\mathscr{W}_i|$ near $z = \alpha$; and so, to the required order of approximation, these contributions are

$$I_3 = e^{-k\alpha}\int_{\alpha-\delta}^{\alpha+\delta}\tilde{\mathscr{W}}_3''dz = -i\,e^{-k\alpha}\left(\frac{\bar{u}'k}{\nu}\right)^{-\frac{1}{3}}\int_{-i\infty}^{+i\infty}\tilde{\mathscr{W}}_3''dZ_3,$$

$$I_2 = e^{-m\alpha}\int_{\alpha-\delta}^{\alpha+\delta}\tilde{\mathscr{W}}_2''dz = -i\,e^{-m\alpha}\left(\frac{\bar{u}'k}{2\nu}\right)^{-\frac{1}{3}}\int_{-i\infty}^{+i\infty}\tilde{\mathscr{W}}_2''dZ.$$

(Note that the prime always denotes d/dz and *not* d/dZ or d/dZ_3.)

Now, $\tilde{\mathscr{W}}_3''$ and $\tilde{\mathscr{W}}_2''$ are given by (7.1) and its counterpart, and we recall that the Wronskian W is a constant whose value is given in result (4.4). Integration by parts yields

$$\int_{-i\infty}^{+i\infty}\tilde{\mathscr{W}}_3''dZ_3 = W^{-1}\left\{\int_{-i\infty}^{Z_3} h_2(\zeta)d\zeta\int_{i\infty}^{Z_3} h_1(\zeta)F_3(\zeta)d\zeta\right.$$
$$\left.-\int_{i\infty}^{Z_3} h_1(\zeta)d\zeta\int_{-i\infty}^{Z_3} h_2(\zeta)F_3(\zeta)d\zeta\right\}_{-i\infty}^{+i\infty} + \int_{-i\infty}^{+i\infty}F_3(Z_3)L(Z_3)\,dZ_3$$
$$= +\int_{-i\infty}^{+i\infty}F_3(Z_3)L(Z_3)\,dZ_3,$$

since the contribution from the end points is zero. Here, we have used the property that the Lommel function $L(Z)$ satisfies equation (4.3). Similarly,

$$\int_{-i\infty}^{+i\infty}\tilde{\mathscr{W}}_2''dZ = \int_{-i\infty}^{+i\infty}F_2(Z)L(Z)\,dZ.$$

Now, $F_3(Z_3)L(Z_3)$ is an analytic function of Z_3 for $-\frac{2}{3}\pi < \arg Z_3 < \frac{2}{3}\pi$, and $F_3(Z_3)L(Z_3)$ behaves asymptotically as Z_3^{-4} for large $|Z_3|$. Contour integration therefore reveals that

$$\int_{-i\infty}^{+i\infty} \tilde{\mathscr{W}}''_3 dZ_3 = 0, \tag{7.2}$$

when $\tilde{\mathscr{W}}''_3$ is approximated by the expression (7.1). This means that I_3 possesses no highest-order viscous term proportional to ν^{-1}.

However, contour integration cannot be employed directly to evaluate I_2: for the function $L(-Z)$ occurs in $F_2(Z)$, and $L(-Z)$ is defined only on the imaginary Z-axis. An integration by parts reveals that

$$I_2 = -\frac{l^2}{m^2\nu}\exp\{-(2m+k)\alpha\}A_3A_1^*\int_{-i\infty}^{+i\infty}\frac{d}{dZ}[L(Z)]\frac{d}{dZ}[L(-Z)]\,dZ.$$

Also, the change of variable $Z = iY$, where Y is real, leads to the result

$$I_2 = \frac{il^2}{m^2\nu}\exp\{-(2m+k)\alpha\}A_3A_1^*\int_{-\infty}^{+\infty}\left\{\left(\frac{dL_r}{dY}\right)^2+\left(\frac{dL_i}{dY}\right)^2\right\}dY,$$

since $L(Z) = L_r(Y)+iL_i(Y)$ and $L(-Z) = L_r(Y)-iL_i(Y)$. Clearly, the integrand is always positive.

Now, from results (4.2a, b), we have

$$L_r\frac{d^2L_r}{dY^2}+L_i\frac{d^2L_i}{dY^2} = -L_r.$$

Therefore, on integrating by parts, we obtain

$$\int_{-\infty}^{+\infty}\left\{\left(\frac{dL_r}{dY}\right)^2+\left(\frac{dL_i}{dY}\right)^2\right\}dY = \int_{-\infty}^{+\infty}L_r\,dY = \mathrm{Im}\int_{-\infty i}^{+\infty i}L(Z)\,dZ.$$

Also, since $L(Z) \sim Z^{-1}$, for $|Z|$ large and $-\frac{2}{3}\pi < \arg Z < \frac{2}{3}\pi$, an appropriate contour integration yields

$$\int_{-\infty i}^{+\infty i}L(Z)\,dZ = \pi i.$$

The final expression for I_2 is therefore

$$I_2 = \frac{\pi i l^2}{m^2\nu}\exp\{-(2m+k)\alpha\}A_3A_1^*, \tag{7.3a}$$

and, by symmetry, the corresponding contribution for wave 1 is

$$I_1 = \frac{\pi i l^2}{m^2\nu}\exp\{-(2m+k)\alpha\}A_3A_2^*. \tag{7.3b}$$

Unlike I_3, both I_1 and I_2 are proportioned to ν^{-1}.

8. The growth rates

The free-surface boundary conditions for waves 1 and 2 involve the expressions (6.3b); and it is shown in the preceding section that the integrals possess contributions I_1 and I_2 from the vicinity of the critical layer, which are $O(\nu^{-1}A_3A_2^*)$

and $O(\nu^{-1}A_3A_1^*)$ respectively. The rest of the range of integration yields contributions which are $O(\omega^{-1}k^2A_3A_2^*)$ and $O(\omega^{-1}k^2A_3A_1^*)$, and these are of the same order of magnitude as the second-order inertia terms at the free surface. Since $\omega/k^2\nu$ is large, a first approximation is obtained by neglecting all but the dominant terms which arise from near the critical layer, and which explicitly involve the liquid viscosity ν. Therefore, from results (7.2) and (7.3*a*, *b*), the growth rates of the waves are, to good approximation,

$$k^{-1}\dot{A}_1 = \kappa A_3A_2^*, \quad k^{-1}\dot{A}_2 = \kappa A_3A_1^*, \quad k^{-1}\dot{A}_3 = o(\omega/\nu k^2)A_1A_2, \qquad (8.1\,a,b,c)$$

where
$$\kappa = \frac{\pi\omega^2 l^2 \exp\{-(2m+k)\alpha\}}{2m^2k\nu(k\bar{u}'-2m\omega)}, \quad \alpha = \omega/k\bar{u}', \qquad (8.2)$$

and ω, $\bar{u}'$, k and m are related by (2.4*a*, *b*) and (3.1). In terms of the angle of propagation θ of the pair of oblique waves,

$$\kappa = \frac{\pi\omega}{k^2\nu}\left(\frac{\cos\theta\sin^4\theta}{\cos^2\theta+2\cos\theta-2}\right)\exp\left\{\frac{\sin^2\theta(1+\cos\theta)}{\cos\theta(\frac{1}{2}\cos\theta-1)}\right\}.$$

This expression reveals that, for given values of $\bar{u}'$ and k, the resonant interaction is likely to be stronger for the smaller than for the larger permissible value of θ (see figure 2). For example, κ equals $-\frac{3}{8}\pi(\omega/k\nu)e^{-3}$ when $\theta = 60°$, but it tends to zero as θ approaches 90°.

It is apparent from results (8.1) that, except possibly when A_1 and A_2 are both very large compared with A_3, the rates of change of A_1 and A_2 are much greater than that of A_3. Accordingly, for most purposes, A_3 may be regarded as a constant; and, in this case, A_1 and A_2 satisfy equations of the form

$$\ddot{A}_j = |\kappa A_3|^2 A_j, \quad |\kappa A_3|^2 \text{ constant} \quad (j = 1, 2).$$

Depending on the initial values of A_1, A_2 and their derivatives, A_1 and A_2 either both grow or both decay exponentially. In particular, if there exists a two-dimensional wave A_3 of small but finite amplitude, and if the inequality (3.2) is satisfied, initially infinitesimal oblique waves which complete the resonant triad may grow to finite amplitude, while the original wave remains essentially unchanged. These waves grow as $\exp\{|\kappa A_3|t\}$, and $|\kappa A_3|$ is typically $O(\omega^2|a_3|/k\nu)$, where a_3 $(= -i\omega^{-1}A_3)$ is the amplitude of the two-dimensional wave. The growth rates are therefore $O(\omega R_a)$, where R_a is the Reynolds number based upon the phase velocity and amplitude of the two-dimensional wave.

Because of condition (6.1), the present second-order analysis holds only when $R_a \ll 1$, and this generally requires the wave amplitude $|a_3|$ to be rather small. However, if the magnitude of A_1 is similar to that of A_2, condition (6.1) does not place such a severe restriction on the amplitudes of the oblique waves (see § 10).

The above analysis has shown that initially small oblique waves may grow until they are large compared with the original two-dimensional wave: but it is not yet established whether, on the basis of second-order theory, the oblique waves may grow *indefinitely* large compared with the two-dimensional wave, for small but non-zero values of the liquid viscosity. More precisely, if $\dot{A}_3 = \mu A_1A_2$ where μ is some constant (which is small compared with κ), changes in A_3 may be ignored only if $|A_1/A_3|$ and $|A_2/A_3|$ are small compared with $|\kappa/\mu|$. When A_1 and A_2 are sufficiently large (but still small enough for second-order theory to apply), it is conceivable that a quasi-equilibrium state might be reached in which the total wave energy is either constant or periodic in time. In order to explore this possibility, the growth of the two-dimensional wave must be examined.

9. The equation for $\dot{A}_3$

The equation for $\dot{A}_3$ may be derived from a purely inviscid analysis, and this has in fact been done. However, although the essential features are quite straightforward, the algebraic details are rather complicated (for example, all the inertia terms in the free-surface boundary condition must be retained). Accordingly, it does not seem worth while to describe the analysis in full: instead, the following brief account should suffice.

Equation (7.1) shows that, near $z = \alpha$, the approximate solution $\tilde{\mathscr{W}}_3''$ *is an* analytic function of Z_3 for $-\frac{2}{3}\pi < \arg Z_3 < \frac{2}{3}\pi$, and that $\tilde{\mathscr{W}}_3''$ decays like Z_3^{-4} as $|Z_3|$ becomes large. This restriction on the argument of Z_3 requires that

$$-\tfrac{7}{6}\pi < \arg(z-\alpha) < \tfrac{1}{6}\pi;$$

and, for this range in argument of $(z-\alpha)$, the viscous solutions tend asymptotically to the inviscid solutions away from the critical layer $z = \alpha$. Accordingly, in evaluating the integral of (6.3*a*), it is permissible to use the inviscid solution for $\tilde{\mathscr{W}}_3$, provided the path of integration is indented *under* the resultant singularity at $z = \alpha$. However, as seen above, this procedure cannot be adopted in evaluating the integrals of (6.3*b*); for these it is not permissible to depart from the real z-axis, since $F_2(Z)$ and the corresponding function for wave 1 are defined only for pure imaginary values of Z.

To second order, the inviscid equation (5.1*b*) yields

$$\tilde{\mathscr{W}}_3''-k^2\tilde{\mathscr{W}}_3 = \frac{4il^2}{m^2k\bar{u}'}e^{-2mz}(z-\alpha)^{-2}\left\{(z-\alpha)^{-2}+\frac{2l^2}{m^2}[m+(z-\alpha)^{-1}]^2\right\}A_1A_2$$

and this result may be used to evaluate the integral of (6.3*a*) on indenting under the singularity at $z = \alpha$. This leads to a result of the form

$$[\tilde{\mathscr{W}}_3'+k\tilde{\mathscr{W}}_3]_{z=0} = \frac{\lambda k^2}{\omega}A_1A_2,$$

where λ is a *complex* constant whose real part derives from the contribution to the integral of the logarithmic singularity at $z = \alpha$. The value of λ does not depend on the viscosity ν, and $|\lambda|$ is typically $O(1)$.

Now, the appropriate second-order equation for $\dot{A}_3$ is (cf. 6.2*a*)

$$ik(2\omega-\bar{u}')\dot{A}_3 = \omega^2(\tilde{\mathscr{W}}_3'+k\tilde{\mathscr{W}}_3)_{z=0}+i\Lambda\omega k^2A_1A_2,$$

where the term in Λ derives from the second-order inertia terms which enter the boundary condition at the free surface, and which were omitted from equations

(8.1). It is easily verified that Λ is real and typically $O(1)$. The final equation for $\dot{A}_3$ is therefore of the form

$$k^{-1}\dot{A}_3 = \gamma A_1 A_2, \tag{9.1}$$

where γ is complex and typically $O(1)$.

On writing $\tau = kt$, $B_{1,2} = \kappa^{\frac{1}{2}}|\gamma|^{\frac{1}{2}}A_{1,2}$, $B_3 = \kappa A_3$, where κ is the (large) real constant defined in (8.2), the complete set of second-order equations (8.1*a*, *b*) and (9.1) become

$$\frac{dB_1}{d\tau} = B_2^* B_3, \quad \frac{dB_2}{d\tau} = B_1^* B_3, \quad \frac{dB_3}{d\tau} = e^{i\phi}B_1 B_2,$$

where $e^{i\phi} = \gamma/|\gamma|$.

It may be shown that

$$\frac{d}{d\tau}|B_1|^2 = \frac{d}{d\tau}|B_2|^2 = 2\,\mathrm{Re}\,(B_1 B_2 B_3^*),$$

$$\frac{d}{d\tau}|B_3|^2 = 2\,\mathrm{Re}\,(e^{i\phi}B_1 B_2 B_3^*),$$

$$\frac{d^2}{d\tau^2}|B_1|^2 = \frac{d^2}{d\tau^2}|B_2|^2 = 2\cos\phi|B_1|^2|B_2|^2 + 2|B_3|^2(|B_1|^2 + |B_2|^2),$$

$$\frac{d^2}{d\tau^2}|B_3|^2 = 2|B_1|^2|B_2|^2 + 2\cos\phi|B_3|^2(|B_1|^2 + |B_2|^2).$$

Now, unless $e^{i\phi} = -1$, it is always possible to choose initial values of B_i such that

$$\frac{d}{d\tau}|B_i|^2 > 0 \quad (i = 1, 2, 3), \quad \tau = 0.$$

Also, when $\cos\phi \geqslant 0$, it is clear that

$$\frac{d^2|B_i|^2}{d\tau^2} \geqslant 0 \quad (i = 1, 2, 3)$$

at all times. Therefore, if $\cos\phi \geqslant 0$, there exists a triad of waves each of which grows indefinitely large with time. If $-1 < \cos\phi < 0$, there may not be three such waves; but in this case it is easily shown that

$$\frac{d^2}{d\tau^2}(|B_{1,2}|^2 + |\cos\phi|\,|B_3|^2) = 2(1-\cos^2\phi)|B_3|^2(|B_1|^2 + |B_2|^2) \geqslant 0.$$

Initial values $B_{1,2,3}$ may therefore be found such that the quantities

$$|B_{1,2}|^2 + |\cos\phi|\,|B_3|^2$$

grow indefinitely with time. It follows that, except when $\cos\phi = -1$ (which case is excluded from the present analysis since γ is complex), the energy of suitable initially small disturbances grows without bound, on the basis of second-order theory.

The above remarks apply to disturbances with initial phases such that

$$d(|B_i|^2)/d\tau > 0 \quad (i = 1, 2, 3).$$

However, it is also possible to choose disturbances such that initially

$$d(|B_i|^2)/d\tau < 0.$$

At first, such disturbances *lose* energy to the mean flow; and, subsequently, either all three waves decay to zero, or the real parts of

$$(B_1 B_2 B_3^*) \quad \text{and} \quad (e^{i\phi} B_1 B_2 B_3^*)$$

change sign and the disturbances grow again.

10. Discussion

The above results show that the primary shear flow is unstable to small, but *finite*, disturbances provided the resonance condition (3.2) is satisfied. The instability mechanism is essentially non-linear, the flow being neutrally stable on the basis of a linearized inviscid analysis. The resonant interaction of a suitable triad of gravity waves extracts energy from the primary flow, most of the energy-transfer taking place in the vicinity of the critical layer, where viscous forces are dominant, and most of the energy going into the oblique waves.

In practice, of course, viscosity also has a damping role due to the weak periodic boundary layers just inside the liquid surface which are associated with each wave. However, it is easily demonstrated that this damping mechanism is likely to be negligible in the present context. The damping effect contributes additional linear terms on the right-hand sides of (8.1*a*, *b*) and (9.1) which are at most $O\{(\omega\nu)^{\frac{1}{2}}A_i\}$ where $i = 1, 2, 3$ respectively. Terms of this magnitude occur in the presence of surface contamination (see Miles 1967; Craik 1968); but, for a clean surface, the appropriate terms are only $O(k\nu A_i)$.

If the oblique waves denoted by A_1 and A_2 are of similar magnitude, viscous damping terms of $O\{(\omega\nu)^{\frac{1}{2}}A_j\}$ $(j = 1, 2)$ are negligible compared with the respective non-linear terms of equations (8.1*a*, *b*) provided

$$\frac{|A_3|k}{\omega}\left(\frac{\omega}{k^2\nu}\right)^{\frac{3}{2}} \gg 1,$$

and the corresponding damping term for A_3 is small compared with the non-linear term of (9.1) if

$$\frac{|A_1|\,|A_2|k}{|A_3|\omega}\left(\frac{\omega}{k^2\nu}\right)^{\frac{1}{2}} \gg 1.$$

On recalling that $A_i = -i\omega a_i$, where a_i is the complex amplitude of the ith wave, the above conditions may be rewritten as

$$|a_3|k\left(\frac{\omega}{k^2\nu}\right)^{\frac{3}{2}} \gg 1, \quad \frac{|a_1|\,|a_2|k}{|a_3|}\left(\frac{\omega}{k^2\nu}\right)^{\frac{1}{2}} \gg 1. \qquad (10.1\,a, b)$$

Also, when $|a_1/a_2|$ is $O(1)$, condition (6.1) is satisfied if

$$|a_3|k\left(\frac{\omega}{k^2\nu}\right) \ll 1, \quad \frac{|a_1|\,|a_2|k}{|a_3|} \ll 1. \qquad (10.2\,a, b)$$

When $\omega/k^2\nu$ is sufficiently large, it is clear that there exist ranges of amplitudes $|a_i|$ which satisfy conditions (10.1*a*) and (10.2*a, b*). However, conditions (10.1*b*) and (10.2*a*) are incompatible unless $|a_1|$ and $|a_2|$ are very large compared with $|a_3|$. Fortunately, this is not a serious restriction; for, when all three amplitudes are of similar magnitude, the growth (or decay) rate of a_3 is insignificant compared with that of a_1 and a_2, and condition (10.1*b*) is unimportant. The growth rate of a_3 is significant only when a_1 and a_2 are large compared with a_3, and then condition (10.1*b*) is likely to be satisfied. With a clean surface, when the viscous damping terms are only $O(k\nu A_i)$, the appropriate conditions are less stringent than (10.1*a*, *b*).

Condition (10.2*a*) limits the validity of the analysis to cases where the amplitude of the two-dimensional wave is rather small: this is because of the remarkable strength of the resonant interaction. The growth rates of the oblique waves are small compared with their frequencies only if the Reynolds number R_a based on the wave amplitude $|a_3|$ and the wave velocity ω/k is small compared with unity. This is a much stronger condition than the more usual one that the maximum wave slope $|a_3 k|$ should be small.

The above analysis is restricted to resonant triads composed of a single two-dimensional wave and two oblique waves which propagate at equal and opposite angles to the primary flow. It is probable that resonance may also occur among triads which do not have this geometrical symmetry: however, the interactions associated with such resonance are likely to be less intense than those discovered here. The reason for this is that, with an asymmetrical resonant triad, there will generally be three *separate* critical layers, one of which is associated with each wave; whereas, in the present case, all three critical layers *coincide*, and the importance of viscous forces is thereby increased.

Throughout the analysis, the effects of surface tension have been ignored. This is justifiable provided all the relevant waves are sufficiently long; but, for shorter waves, the appropriate resonance condition would be much modified from that given in (3.2). However, if a resonant triad of gravity-capillary waves were found, it seems likely that the same non-linear energy-transfer mechanism would operate.

As mentioned in the introduction, the resonance condition (3.2) is rather severe, requiring larger velocity gradients at typical wave-numbers than might be expected to occur in nature. For example, for fairly long gravity waves with $k \sim O(10^{-2})\,\text{cm}^{-1}$, condition (3.2) requires that the velocity must change by at least $O(10)\,\text{cm sec}^{-1}$ for every centimetre of depth. However, it is shown in the next section that the condition for resonance may be more easily met for gravity waves on the interface between two fluids of different densities. Also, the characteristic wave velocities are typically smaller for interfacial waves than for surface waves; and the restrictions corresponding to (10.1*a*, *b*) and (10.2*a*, *b*) on the validity of an interfacial-wave theory allow larger wave amplitudes than are permissible for surface gravity waves.

11. The theory for interfacial waves

Let the two fluids have densities ρ_1 and ρ_2 ($< \rho_1$), and let their velocity gradients be $\bar{u}_1'$ and $\bar{u}_2'$, where the subscripts 1 and 2 refer, respectively, to the lower and upper fluids. The linear dispersion relations for interfacial gravity waves which correspond to (2.4*a, b*) are easily shown to be

$$\omega^2 - \bar{u}_*'\omega - g_*k = 0, \quad \omega_0^2 - \frac{k}{2m}\bar{u}_*'\omega_0 - g_*m = 0, \tag{11.1a, b}$$

where
$$g_* = \frac{g(\rho_1-\rho_2)}{\rho_1+\rho_2}, \quad \bar{u}_*' = \frac{\rho_1\bar{u}_1'-\rho_2\bar{u}_2'}{\rho_1+\rho_2}.$$

It is clear that the above equations are identical with (2.4*a, b*) with g and $\bar{u}'$ replaced by g_* and $\bar{u}_*'$. The appropriate resonance condition for interfacial waves is therefore

$$\frac{\bar{u}_*'}{(g_*k)^{\frac{1}{2}}} = \frac{(\rho_1\bar{u}_1'-\rho_2\bar{u}_2')}{(gk)^{\frac{1}{2}}(\rho_1+\rho_2)(\rho_1-\rho_2)} \geqslant \frac{7+48^{\frac{1}{2}}}{(8+48)^{\frac{1}{2}}} \tag{11.2}$$

by comparison with (3.2). In particular, when the lower fluid is only slightly heavier than the upper, and when $\rho_1\bar{u}_1'$ is not very close to $\rho_2\bar{u}_2'$, condition (11.2) is satisfied even by rather small velocity gradients, provided $\rho_1\bar{u}_1' > \rho_2\bar{u}_2'$.

The non-linear analysis described in §§5–8 may readily be extended to deal with interfacial gravity waves. Since the influence of viscosity in the vicinity of the critical layer dominates the non-linear interaction process, equations (8.1*a, b, c*) remain valid, but ω is now given by the dispersion relation (11.1*a*), while ν and $\bar{u}'$ are the kinematic viscosity and velocity gradient of that fluid in which the critical layer occurs. Additional complications arise in determining the appropriate equation for $\acute{A}_3$; but, in any case, this hardly seems worth while since the oblique waves again grow much more rapidly than the two-dimensional one.

The fact that condition (11.2) is readily satisfied for two fluids of slightly different densities and dissimilar velocity gradients suggests that a fairly simple experiment might be devised to confirm (or disprove!) the existence of the non-linear instability mechanism examined above.

I am grateful to Dr R. E. Kelly, Dr R. E. Davis, Professor J. W. Miles and Dr H. Huppert, with whom I had numerous helpful discussions.

This work was performed during a visit to the University of California, Institute of Geophysics and Planetary Physics, La Jolla, California. The visit was supported by the National Science Foundation, by the Office of Naval Research, and by a travel grant from the Sir James Caird Trust.

REFERENCES

Benjamin, T. B. & Fier, J. E. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417.

Benney, D. J. 1961 A non-linear theory for oscillations in a parallel flow. *J. Fluid Mech.* **10**, 209.

BENNEY, D. J. 1964 Finite amplitude effects in an unstable laminar boundary layer. *Phys. Fluids*, **7**, 319.

CRAIK, A. D. D. 1968 Wind-generated waves in contaminated liquid films. *J. Fluid Mech.* **31**, 141.

DAVIS, R. E. & ACRIVOS, A. 1968 The stability of oscillatory internal waves. *J. Fluid Mech.* **30**, 723.

KELLY, R. E. 1968 On the resonant interaction of neutral disturbances in inviscid shear flows. *J. Fluid Mech.* **31**, 789.

MILES, J. W. 1967 Surface-wave damping in closed basins. *Proc. Roy. Soc.* A **297**, 459.

PHILLIPS, O. M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part 1. The elementary interactions. *J. Fluid Mech.* **9**, 193.

RAETZ, G. S. 1959 A new theory of the cause of transition in fluid flows. *Norair Rept.* NOR–59–383. Hawthorne, California.

STUART, J. T. 1962 On three-dimensional non-linear effects in the stability of parallel flows. *Advanc. Aero. Sci.* **3**, 121.